REFERENCES

1. Golubev, V. V., Lectures on the Integration of the Equations of Motion of a Heavy Rigid Body Around a Fixed Point. Moscow, Gostekhizdat, 1953.
2. Kolmogorov, A. N., General theory of dynamic systems and classical mechanics. In: Proc. International Congress of Mathematicians, Amsterdam, 1954. Moscow, Fizmatgiz, 1961.
3. Tatarinov, Ia. V., On the investigation of the phase topology of compact configurations with symmetry. Vestn. Moskovsk. Gos. Univ., Ser. Matem., Mekhan., № 5, 1973.
4. Arnol'd, V. I., On a theorem of Liouville concerning integrable problems of dynamics. Sibirsk. Matem. Zh., Vol. 4, № 2, 1963.
5. Kozlov, V. V., Geometry of the "action-angle" variables in the Euler-Poinsot problem. Vestn. Moskovsk. Gos. Univ., Ser. Matem., Mekhan., № 5, 1974.

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ON QUANTITATIVE ESTIMATES OF THE INDETERMINACY OF MOTION

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We propose quantitative estimates of the indeterminacy of the prediction of the motion of a controlled system described by ordinary differential equations.

1. The motion of a controlled system is specified by the equation

$$dx / dt = F(t, x, u(t)), \quad t \in I, \qquad I = \{t : t_0 \leqslant t < t_*\} \qquad (1.1)$$

where $x = (x_1, \ldots, x_n)$ and $u = (u_1, \ldots, u_r)$ are vectors in real n- and r-dimensional spaces R_x^n and R_u^r, respectively, t is time, t_0 is a number, t_* is either a number or the symbol ∞. The norm $\|x\| = \max_i |x_i|$ is defined in space R_x^n. The vector $x = x(t)$, $t \in I$ characterizes the state of the controlled system, $u = u(t)$, $t \in I$ is an input whose graph $\omega = \{(t, u): u = u(t), t \in I\}$ belongs to an admissible set $\Omega = \{\omega\}$, $\omega\, [t_0, t]$ is the restriction of ω onto $[t_0, t] \cap I$.

Suppose that a solution of system (1.1), starting on a given open set $V_0 \subset R_x^n$, exists for all $t \in I$ at arbitrarily chosen $\omega \in \Omega$: $x(t) = \varphi(t, t_0, x_0, \omega\,[t_0, t])$, $t \in I$, is any such solution, where

$$x(t_0) = x_0 \in V_0 \qquad (1.2)$$

Let $S_\delta(x_0) = \{b_0 : \|b_0 - x_0\| \leqslant {}^1/_2\, \delta\}$, $\delta > 0$ be a ball which characterizes the region of admissible initial states of system (1.1), if the possible measurement errors of the controlled system's initial state are taken into account. At an instant $t \in I$ we consider the set S_{ρ_t} of states of system (1.1) on all possible motions of it (on the graphs of the solutions in $I \times R_x^n$) starting from $S_\delta(x_0)$ for a specified $\omega \in \Omega$. In other words, $S_{\rho t} = \Phi_t(S_\delta(x_0))$ is the image of the ball $S_\delta(x_0)$, where Φ_t is a mapping from R_x^n into R_x^n considered as the transition operator from x_0 to x, Φ_t: $x = \varphi\{\cdot\}$, $x_0 \in V_0$, whereas $\rho_t > 0$ is the diameter of set S_{ρ_t}, while the abbreviation $T\ \{\cdot\}$ is used here and elsewhere to denote the functional $T\ (t,\ t_0,\ x_0,\ \omega\ [t_0,\ t])$.

As a quantitative measure of the local divergence of the motions of system (1.1), starting in an arbitrarily small neighborhood of a given point $x\ (t_0) = x_0$, we consider the quantity

$$\lambda\{\cdot\} = \lim_{\delta\to 0}(|S_{\rho_t}| / |S_\delta(x_0)|) \tag{1.3}$$

where $|S_{\rho_t}|$ and $|S_\delta(x_0)|$ are the volumes of sets S_{ρ_t} and $S_\delta(x_0)$, respectively. With due regard to equality (1.5) from [1], relation (1.3) can be written as

$$\lambda\ \{\cdot\} = \exp\ L\ \{\cdot\} \tag{1.4}$$

$$L\{\cdot\} = \int_{t_0}^{t} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_i(\tau, \varphi(\tau, t_0, x_0, \omega[t_0, \tau]), u(\tau))\, d\tau \tag{1.5}$$

Here $L\ \{\cdot\}$ is a functional on the motions of system (1.1), which in [1] is called the degree of lability of system (1.1), $F\ (t,\ x,\ u\ (t))$, $(t,\ x) \in A$, is a continuously differentiable function of t, x for any $\omega \in \Omega$, where

$$A = \{(t,\ x):\ x = \varphi\ \{\cdot\},\ t \in I,\ x_0 \in V_0,\ \omega \in \Omega\}$$

The numerical value of the degree of lability of system (1.1) can be obtained, obviously, by integrating the system of equations with initial conditions

$$\frac{d\varphi\{\cdot\}}{dt} = F(t, \varphi\{\cdot\}, u(t)), \quad \frac{dL\{\cdot\}}{dt} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_i(t, \varphi\{\cdot\}, u(t)), \quad t \in I \tag{1.6}$$

$$\varphi\{\cdot\}|_{t=t_0} = x_0, \quad L\{\cdot\}|_{t=t_0} = 0$$

The set of ordered pairs $(\varphi\ \{\cdot\},\ L\ \{\cdot\})$, on the one hand, determines a motion of system (1.1), on the other hand, characterizes the degree of indeterminacy, caused principally by the unavoidable errors in the initial state of prediction of the motion of the controlled system by means of the system of differential equations (1.1). We assume that system (1.6) can be integrated by any available method (for example, numerically on a computer).

2. The quality of performance of a controlled system is frequently judged by a scalar function $y = y\ (t)$, $t \in I$, called the output of system (1.1) and related to the state vector $x = x\ (t)$, $t \in I$, by the finite equation

$$y\ (t) = \Phi\ (t,\ x\ (t)) \tag{2.1}$$

We assume that $\Phi\ (t,\ x)$, $(t,\ x) \in A$ is a continuously differentiable function of t, x. In order to estimate the influence of the inaccuracy in specifying $x\ (t_0) = x_0$ on the quantity $y = y\ (t)$, $t \in I$, we introduce a special quantitative measure. To do this we first implement the following auxiliary constructions.

We consider the ball $S_\delta\ (x_0) = \{b_0 : \|\ b_0 - x_0\ \| \leqslant {}^1/_2\delta\}$, $\delta > 0$, simulating the errors in the initial state $x\ (t_0) = x_0 \in V_0$. We select the positive number δ from the condition $S_\delta\ (x_0) \subset V_0$. Let

$$O_t:\ y = \Phi\ (t,\ \varphi\ (t,\ t_0,\ x_0,\ \omega\ [t_0,\ t])),\quad x_0 \in V_0 \tag{2.2}$$

be a mapping from space R_x^n into the one-dimensional space $R_y^1 = \{y\}$, and let D_t be the image of ball $S_\delta(x_0)$ in space R_y^1, induced by the mapping O_t, $D_t = O_t(S_\delta(x_0))$. It is evident that (2.2) can be treated as an n-dimensional hypersurface in the $(n+1)$-dimensional space $R_x^n \times R_y^1 = \{(x, y)\}$, and D_t as a part of this hypersurface, corresponding to the region $S_\delta(x_0)$. The area $|D_t|$ of this hypersurface is

$$|D_t| = \int_{S_\delta(x_0)} \sqrt{1+\varkappa^2\{\cdot\}}\, dx_{10} \ldots dx_{n0} \tag{2.3}$$

$$\varkappa\{\cdot\} = \left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n} \frac{\partial\Phi(\cdot)}{\partial x_j}\frac{\partial\varphi_j\{\cdot\}}{\partial x_{k_0}}\right)^2\right)^{1/2} \tag{2.4}$$

if the quantity $\varkappa\ \{\cdot\}$ takes positive values.

In order to estimate the influence of the inaccuracy in specifying the initial state $x(t_0) = x_0$ of system (1.1) on the accuracy of determining the current value of its output $y = y(t)$, $t \in I$, we consider the quantity

$$\sigma = \sigma(t, t_0, x_0, \omega[t_0, t]) = \lim_{\delta\to 0}(|D_t| / |S_\delta(x_0)|) \tag{2.5}$$

where $|S_\delta(x_0)|$ is the volume of set $S_\delta(x_0)$. By applying the theorem on the mean to (2.3), and next substituting the resulting relation into (2.5), we have $\sigma = \sqrt{1+\varkappa^2}$, $\varkappa > 0$; whence

$$\varkappa = \sqrt{\sigma^2 - 1}, \quad \sigma^2 > 1 \tag{2.6}$$

We denote

$$l\{\cdot\} = \ln\varkappa\{\cdot\} \tag{2.7}$$

From (2.6), (2.7) follows the equivalence of the statements:

a) $\sigma\{\cdot\} \to \infty \Leftrightarrow l\{\cdot\} \to \infty$, b) $\sigma\{\cdot\} \to 1 \Leftrightarrow l\{\cdot\} \to -\infty$

Therefore, to estimate the accuracy of determining the output $y = y(t)$, $t \in I$ we can use the quantity $l\{\cdot\}$ defined by (2.4), (2.7) instead of the quantitative measure $\sigma\{\cdot\}$. This quantity is called the degree of lability of output y of system (1.1). The numerical values of $l\{\cdot\}$ can be found from formulas (2.4), (2.7), allowing for the fact that $\varphi_i\{\cdot\}$ and $\partial\varphi_i\{\cdot\} / \partial x_{j0}$, $i, j = 1, \ldots, n$, satisfy, according to Peano's theorem [2], the system of equations

$$\frac{d\varphi\{\cdot\}}{dt} = F(t, \varphi\{\cdot\}, u(t)) \tag{2.8}$$

$$\frac{d}{dt}\frac{\partial\varphi\{\cdot\}}{\partial x_0} = \frac{\partial}{\partial x}F(t, \varphi\{\cdot\}, u(t))\frac{\partial\varphi\{\cdot\}}{\partial x_0}$$

$$\varphi\{\cdot\}|_{t=t_0} = x_0, \quad (\partial\varphi\{\cdot\} / \partial x_0)|_{t=t_0} = E_n, \quad t \in I$$

Here $\partial\varphi\ \{\cdot\} / \partial x_0$ is the $(n \times n)$-matrix of partial derivatives $\partial\varphi_i\{\cdot\} / \partial x_{j0}$, $i, j = 1, \ldots, n$; $\partial F(\cdot) / \partial x$ is the $(n \times n)$-matrix of partial derivatives $\partial F_i(\cdot) / \partial x_j$ considered on the solution $x(t) = \varphi\{\cdot\}$, $t \in I$, E_n is the unit $(n \times n)$-matrix. We assume that system (2.8) can be integrated by any available method (for example, numerically on a computer).

For any $t \in I$ the quantity $l\{\cdot\}$ (and $L\{\cdot\}$) is a continuous function of the initial state $x(t_0) = x_0 \in V_0$. Consequently, if V_0^* is a compactum belonging to V_0, then by numerical methods we can find initial points x_0^1 and x_0^2, belonging to this compactum, such that the quantity $l\{\cdot\}$ (or $L\{\cdot\}$), considered at some given instant $\xi \in I$, reaches its greatest lower bound and least upper bound, respectively.

3. The motion of an airplane at the straight-and-level flight, stabilized at a constant altitude $H = H_0$ by elevator deflections as necessary, is given by the equations [3]

$$m_0 dv_* / dt = R \cos(\alpha - \nu) = Q, \qquad Y + R \sin(\alpha - \nu) = G_0 \tag{3.1}$$

$$Q = C_x \frac{\rho_0 v^2}{2} S, \quad Y = C_y \frac{\rho_0 v^2}{2} S, \quad t \in I$$

Here $v_* = v_*(t)$ is the airplane's ground speed (speed relative to the Earth's surface) at instant $t \in I = \{t\colon 0 \leqslant t < t_*\}$, R is the thrust of the engine, Q is the drag, Y is the lift, α is the airplane's angle of attack, ν is the angle between the engine thrust direction and the fuselage axis, m_0 and G_0 are the airplane's mass and gravity, respectively, C_x and C_y are drag and lift coefficients, taking positive values only, ρ_0 is the air density at altitude H_0, S is the wing area, v is the airplane's air speed or flying speed (speed relative to the atmosphere, masses of which can move relative to the Earth's surface). The ground and air speeds are related by the equation

$$v_* = v + w \tag{3.2}$$

where w is the wind speed in the horizontal direction; $w > 0$ for a tail wind, $w < 0$ for a head wind.

The peculiarities of the airplane's behavior in a turbulent atmosphere, caused by wind gust conditions, are of essential significance. By investigating the influence of a single horizontal wind gust on the motion of the airplane at the straight-and-level flight, we assume

$$w = \begin{cases} a, & t \geqslant 0 \\ 0, & t < 0 \end{cases} \tag{3.3}$$

where a is a number characterizing the wind gust intensity. Then, from (3.2), (3.3) it follows that

$$dv_* / dt = dv / dt, \quad t \in I \tag{3.4}$$

For turbojet engines, at low flying speeds (for example, at the takeoff and landing operating conditions we can assume [3]

$$R = h_0 - h_1 v \tag{3.5}$$

where h_0 and h_1 are positive numbers. The quantity h_0 is a function of the angle of rotation s of the engine thrust control lever. At low flying speeds when the compressibility of the air can be neglected, the coefficients C_x and C_y satisfy the equation [3]

$$C_x = C_{x0} + BC_y^2 \tag{3.6}$$

where C_{x0} and B are positive coefficients practically independent of the flight Mach number.

By examining Eqs. (3.1) – (3.6) simultaneously under the condition that at the airplane's straight-and-level flight the engine thrust component in the second equation of system (3.1) is negligibly small and that the angle $(\alpha - \nu)$ between the engine axis and the tangent to the flight path is not large, we obtain

$$dv / dt = c(s) - dv - ev^2 - f / v^2 \equiv P(s, \; v), \quad t \in I \tag{3.7}$$

$$c(s) = h_0 s / m_0, \quad d = h_1 / m_0, \quad e = \rho_0 C_{x0} S / 2m_0, \quad f = 2gBG_0 / \rho_0 S$$

where g is the gravitational acceleration. For each fixed position $s = s_\infty$ of the thrust control lever the air speed v_∞ of uniform straight flight can be found from the condition

$$P(s_\infty, \; v_\infty) = 0. \tag{3.8}$$

Thus, the positive values of the roots of Eq. (3.8), having a physical meaning, depend upon the quantity s_∞.

Examining the particular case when $h_1 = 0$ and $s = s_\infty$, Painlevé [4] was the first to show that Eq. (3.8) usually has two positive roots $\lambda_1 = (v_\infty)_1$ and $\lambda_2 = (v_\infty)_2$ which determine two possible values of the speed of a uniform straight-and-level flight at a given altitude $H = H_0$, caused by the balancing of the engine thrust and the drag, and this balance can be Liapunov-stable or -unstable (the points λ_1 and λ_2 in Fig. 1).

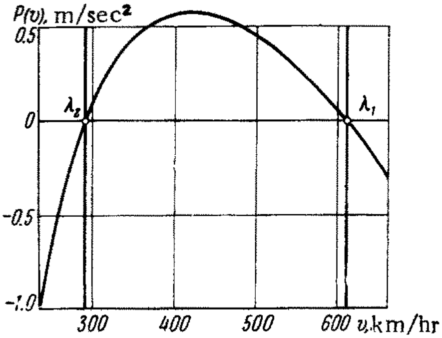

Fig. 1

Let us assume that the airplane, being for all $t < 0$ at a steady uniform straight-and-level flight in still air, suddenly enters a region of a horizontal airstream whose speed is determined by relation (3.3). In this case $w = 0$ and $v = v_*$ for all $t < 0$ and

$$v(-0) = \lim_{\mu \to 0, \, \mu < 0} v(\mu) = \lambda_i \tag{3.9}$$

and, as seen from (3.2), (3.3) and (3.9), we should assume

$$v(0) = \lambda_i - a = v_0 \tag{3.10}$$

as the initial value of the air speed. From (3.10) it follows that a perturbation of the airplane's initial air speed can be caused not only by a gusty wind but also by a sharp intermittent change of the control lever position s, which in accordance with the equilibrium condition (3.8) leads to a change in the steady air speed $\lambda_i = (v_\infty)_i$.

In order to estimate the sensitivity of the air speed $v(t) = v(t, \; 0, \; v_0)$ at the straight-and-level flight to perturbations of its initial speed $v(0) = v_0$, we determine the degree of lability of Eq. (3.7) from formula (1.5)

$$L(t, 0, v_0) = \int_0^t \left(-d - 2ev^2 + \frac{2f}{v^3} \right) d\tau$$

The quantity $L(t, \; 0, \; v_0)$ characterizes the degree of indeterminacy, caused by perturbations of the initial air speed $v(0) = v_0$ in the variation in time, predicted by Eq. (3.7), of the air speed $v(t) = v(t, \; 0, \; v_0)$. Numerical values of $L(t, \; 0, \; v_0)$ can be found by solving the problem, analogous to (1.6)

$$\begin{aligned} dv/dt &= c(s) - dv - ev^2 - f/v^2 \\ dL/dt &= -d - 2ev + 2f/v^3, \quad t \in I \\ v(0) &= v_0, \quad L(0, \; 0, \; v_0) = 0 \end{aligned} \tag{3.11}$$

In Figs. 2, 3 we have constructed families of curves $v = v(t, \; 0, \; v_0)$, $L = L(t, \; 0, \; v_0)$, $t \in I$, obtained by a numerical integration of Eqs. (3.11) on a computer for different values of the initial air speed $v(0) = v_0$ and for one fixed position s_∞ of the engine thrust control lever. The calculations were made for an hypothetical transport aircraft with the parameters $c(s_\infty) = 2.235$ m/sec², $d = 2 \cdot 10^{-3}$ sec⁻¹, $e = 6.9 \cdot 10^{-5}$ m⁻¹, $f = 1.445 \cdot 10^4$ m³/sec⁴, corresponding to the takeoff-landing mode of the straight-and-level flight. An analysis of the curves, shown in Fig. 3, of the variation of the degree of lability $L = L(t, 0, v_0)$ shows: the less is the airplane initial air speed $v(0) = v_0$ (Fig. 2) the greater is the value of the degree of lability $L = L(t, 0, v_0)$ for arbitrarily chosen $t \in I$; but this, in its own turn, signifies a large indeterminacy if the current value, predicted by Eq. (3.7), of the air speed because of the perturbations of initial air speed $v(0) = v_0$. Consequently, as we see from (3.2), (3.3), (3.10) and Fig. 3, a sharp gust of tail wind must cause a particularly large divergence in the motions of Eq. (3.7).

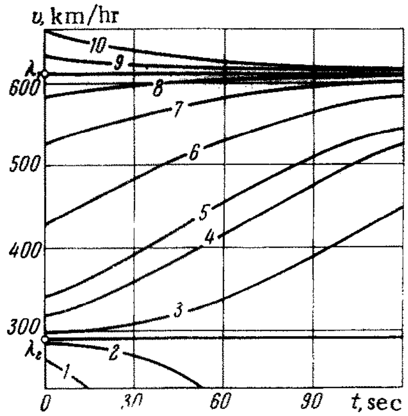

Fig. 2

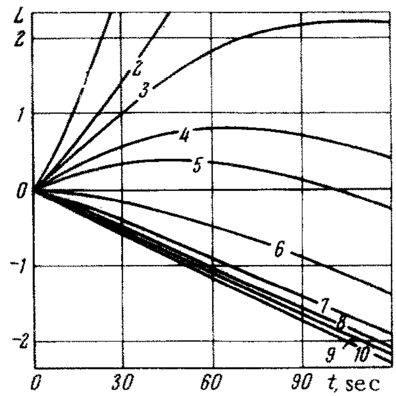

Fig. 3

If we know how the magnitude of the degree of lability $L = L(t, 0, v_0)$ of Eqs. (3.7) varies on some selected motion $v = v(t, 0, v_0)$, $t \in I$, then, obviously, we can answer the question on how a motion neighboring this motion behaves. For example, curve 5 in Fig. 3, characterizing the time variation of the degree of lability of Eq. (3.7), shows that in Fig. 2 the motion neighboring curve 5 moves at first away from this curve (on the time interval where the degree of lability takes positive values) and then approaches it (on the time interval where the degree of lability is negative), whereas the absolute value of the degree of lability, considered at a given instant, characterizes the intensity of the course of all these processes.

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REFERENCES

1. Kostiukovskii, Iu. M. -L., On one idea of Chetaev. PMM Vol. 37, №1, 1973.
2. Hartman, P., Ordinary Differential Equations. New York–London–Sidney, J. Wiley and Sons, Inc., 1964.
3. Ostoslavskii, I. V. and Strazheva, I. V., Flight Dynamics. Aircraft Trajectories, 2nd ed. Moscow, "Mashinostroenie", 1969.
4. Painlevé, P., Étude sur le régime normal d'un aéroplane. Techn. Aéronaut., Vol. 1, pp. 3–11, 1911.

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